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EXTENSION OF A COMPOSITE PLANE WITH A THIN ELASTIC INCLUSION EMERGING ORTHOGONALLY ON A STRAIGHT MATERIAL INTERFACIAL LINE*

A.A. EVTUSHENKO and YU.I. SOROKATYI

A problem of plane elasticity theory concerning the extension of a composite plane with a thin finite elastic inclusion emerging orthogonally on a straight material interfacial line is solved. The results of a numerical solution of the singular integral equation of this problem are represented for the stress intensity factors and the shear stresses on the axial line of the inclusion. Solutions of the corresponding problems for cracks /1-4/ and an absolutely stiff inclusion follow as special cases from the results obtained. Earlier /5-7/, the singularity of the stress field near the vertex of a slightly pliable inclusion emerging on the free boundary of a homogeneous isotropic half-plane was investigated.

The problem of the elastic equilibrium of a piecewise-homogeneous plane consisting of two bonded half-planes within one of which a thin elastic inclusion of finite length $2a_0 = b - a$ and thickness $2h$ was located, was solved in /8/. The composite body is subjected to homogeneous tensile force fields at infinity σ_0, σ_1 and σ_2, where

$$k_0^1\sigma_1 = k_0^2\sigma_2 - (k_1^2 - k_1^1)\sigma_0, \quad k_0^j = (1 + \varkappa_j)/(8\mu_j), \quad k_1^j = (3 - \varkappa_j)/(8\mu_j) \tag{1}$$

where $\varkappa_j = 3 - 4\nu_j$ for plane strain $\varkappa_j = (3 - \nu_j)/(1 + \nu_j)$ for the generalized plane state of stress and $\mu_j, \nu_j\ (j = 1, 2)$ are the shear modulus and Poisson's ratio of the half-plane materials. By using a Mellin transformation, expressions are obtained for the stresses and the derivatives of the displacements on an axis that agrees with the middle line of the inclusion on the segment $a < r < b$ (/8/, formula (10)). Passing to the limit $a \to 0$, $\theta = \pi$ in these relationships (Fig.1), we find

$$\begin{aligned}
&\tau_{1\theta\theta}(r, \pi) = \tau_{1\theta\theta}^0(r, \pi) + a_{11}t_1(r) + a_{12}t_2(r) + h_1(r)\\
&u_{1r}'(r, \pi) = u_{1r}^{0\prime}(r, \pi) + a_{21}t_1(r) + a_{22}t_2(r) + h_2(r),\\
&0 < r < b
\end{aligned} \tag{2}$$

Here

$$t_j(r) = \frac{1}{\pi}\int_0^b \frac{f_j(r_0)}{r_0 - r}\,dr_0, \quad h_i(r) = \frac{1}{\pi}\int_0^b \sum_{j=1}^2 K_{ij}(r_0, r) f_j(r_0)\,dr_0$$

$$K_{ij}(r, r_0) = \sum_{n=0}^2 c_{ij}^n \frac{r^n}{(r_0 + r)^{n+1}} \quad (i, j = 1, 2)$$

$$c_{11}^n = \frac{b_{11}^n}{1 + \varkappa}, \quad c_{12}^n = -a_{12}b_{12}^n, \quad c_{21}^n = \frac{a_{21}b_{21}^n}{\varkappa_1},$$

$$c_{22}^n = \frac{b_{22}^n}{1 + \varkappa_1} \quad (n = 0, 1, 2)$$

$$b_{11}^0 = b_{22}^0 = l_1 + 3(2 + \varkappa_1)l_2, \quad b_{11}^1 = -2(7 + \varkappa_1)l_2,$$

$$b_{12}^0 = l_1 + 3l_2, \quad b_{12}^1 = b_{21}^1 = -12l_2$$

$$b_{21}^0 = l_1 + (4 - \varkappa_1^2)l_2, \quad b_{22}^1 = 4(5 - \varkappa_1)l_2, \quad b_{ij}^2 = 8l_2$$

$$(i, j = 1, 2)$$

$$l_1 = \frac{m\varkappa_1 - \varkappa_2}{2(m + \varkappa_2)}, \quad l_2 = \frac{1 - m}{2(1 + m\varkappa_1)}, \quad m = \frac{\mu_2}{\mu_1}$$

$$a_{11} = a_{22} = \frac{\varkappa_1 - 1}{2(\varkappa_1 + 1)}, \quad a_{12} = \frac{2\mu_1}{\varkappa_1 + 1}, \quad a_{21} = -\frac{\varkappa_1}{2\mu_1(\varkappa_1 + 1)}$$

$f_j(r)\ (j = 1, 2)$ are jumps in the stress and displacement simulating the presence of the inclusion

$$\tau_{1r\theta}(r, \pi) = -\tfrac{1}{2}f_1(r), \quad u_{1r}'(r, \pi) = -\tfrac{1}{2}f_2(r), \quad 0 < r < b \tag{3}$$

$\tau_{1\theta\theta}^0$ and u_{1r}^0 are the stress and displacement due to a given external load when there is no inclusion; the prime denotes the derivatives of the appropriate quantities with respect to the variable r.

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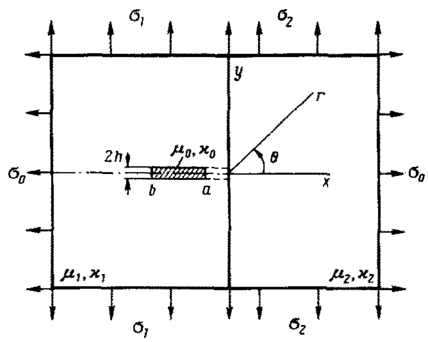

Fig.1

The interaction conditions for a thin elastic inclusion with a matrix, written taking account of the symmetry of the problem under consideration relative to the plane $0 < r < \infty$, $\theta = \pi$, have the form /9-11/

$$
\begin{aligned}
&u_{1r}'(r,\pi) = k_0^0 \tau_{1r}^a(r,\pi) - k_1^0 \tau_{1\theta\theta}(r,\pi) \\
&h^{-1} u_{1\theta}(r,\pi) = k_0^0 \tau_{1\theta\theta}(r,\pi) - k_1^0 \tau_{1r}^a(r,\pi) \\
&u_{1\theta}'(r,\pi) = \mu_0^{-1} \tau_{1r\theta}(r,\pi),\ 0 \leqslant r < b \\
&\tau_{1r}^a(r,\pi) = \tau_{1rr}(0,\pi) - h^{-1}\int_0^r \tau_{1r\theta}(r_0,\pi)\,dr_0; \quad k_0^0 = \frac{1+\varkappa_0}{8\mu_0}, \\
&k_1^0 = \frac{3-\varkappa_0}{8\mu_0}
\end{aligned} \tag{4}
$$

(μ_0, ν_0 are the shear modulus and Poisson's ratio of the material of the inclusion).

Eliminating the mean tensile stress $\tau_{1r}^a(r,\pi)$ in the transverse section of the inclusion from the first two equalities in (4), we obtain

$$
\begin{aligned}
&u_{1r}'(r,\pi) + k_0^0 (hk_1^0)^{-1} u_{1\theta}(r,\pi) = -k_2 \tau_{1\theta\theta}(r,\pi),\ 0 \leqslant r \leqslant b \\
&k_2 = (k_1^{02} - k_0^{02})/k_1^0
\end{aligned} \tag{5}
$$

Subsitution of the expressions (2) into condition (5) taking the third of relations (4) into account results in the singular integral equation (SIE)

$$
\begin{aligned}
&\frac{\lambda_1}{\pi}\int_0^b \frac{f_2(r_0)}{r_0 - r}\,dr_0 + \lambda_2 \int_0^r f_2(r_0)\,dr_0 + \frac{1}{\pi}\int_0^b K(r_0,r) f_2(r_0)\,dr_0 = F(r) \\
&f_1(r) = \mu_0 f_2(r), \quad 0 < r < b \\
&K(r_0,r) = \mu_0 k_2 K_{11}(r_0,r) + k_2 K_{12}(r_0,r) + \mu_0 K_{21}(r_0,r) + \\
&\quad K_{22}(r_0,r) \\
&F(r) = -k_2 \tau_{1\theta\theta}^0(r,\pi) - u_{1r}^{0\prime}(r,\pi) - k_0^0 (k_1^0 h)^{-1} u_{1\theta}(0,\pi) \\
&\lambda_1 = \mu_0 a_{11} k_2 + a_{12} k_2 + \mu_0 a_{21} + a_{22},\ \lambda_2 = -k_0^0 (2hk_1^0)^{-1}
\end{aligned} \tag{6}
$$

The desired function $f_2(r)$ satisfies the additional condition /8, 10/

$$
\begin{aligned}
&\int_0^b f_2(r_0)\,dr_0 = A \\
&A = [u_{1\theta}(0,\pi) - u_{1\theta}(b,\pi)] \cdot h^{-1} \simeq \varepsilon_{1\theta\theta}(0,\pi) - \varepsilon_{1\theta\theta}(b,\pi)
\end{aligned} \tag{7}
$$

(the expression for A is written on the basis of the definition of the jumps (3)).

We represent the deformation $\varepsilon_{1\theta\theta}(0,\pi)$ in the form

$$
\varepsilon_{1\theta\theta}(0,\pi) = k_0^0(\sigma_1 + M) - k_1^0(\sigma_0 + L) \tag{8}
$$

where the constants M and L should assume continuity of the change in the solution of the problem when going from the inclusion material over to the host material. The values

$$
\begin{aligned}
&M = \Delta_1/\Delta,\ L = \Delta_2/\Delta \\
&\Delta = \delta_{0+}^{01}\delta_{0+}^{20} - \delta_{1+}^{01}\delta_{1+}^{02} \\
&\Delta_1 = 2(k_0{}^0 k_1{}^0 - k_0{}^1 k_1{}^2)\sigma_0 + (\delta_{1-}^{01}\delta_{1+}^{02} - k_0{}^0\delta_{0+}^{01})\sigma_1 + k_0{}^2\delta_{0+}^{01}\sigma_2, \\
&\Delta_2 = (\delta_{0-}^{01}\delta_{0+}^{20} + \delta_{1-}^{02}\delta_{1+}^{01})\sigma_0 + (k_0{}^2\delta_{1-}^{01} - 2k_0{}^0 k_1{}^1)\sigma_1 + k_0{}^2\delta_{1+}^{01}\sigma_2 \\
&\delta_{p\pm}^{ln} = k_p{}^l \pm k_p{}^n; \quad p = 0, 1; \quad l, n = 0, 1, 2
\end{aligned} \tag{9}
$$

satisfy this requirement.

It can be seen by direct substitution that when there is no inclusion ($\mu_0 = \mu_1$, $\nu_0 = \nu_1$), the presence of a crack ($\mu_0 = 0$, $\nu_0 = 0$) or an absolutely stiff inclusion ($\mu_0 = \infty$, $\nu_0 = 0.5$) the values $\varepsilon_{1\theta\theta}(0,\pi)$ agree with the solutions of the appropriate problems. The constant $\varepsilon_{1\theta\theta}(b,\pi)$ is obtained from (8) and (9) by replacing all subscripts 2 by 1.

In the case of an absolutely stiff inclusion ($\mu_0 = \infty$, $\nu_0 = 0.5$) $f_2(r) = 0$ follows from the integral Eq.(6), and we obtain the equation

$$
\begin{aligned}
&\frac{a_{21}}{\pi}\int_0^b \frac{f_1(r_0)}{r_0 - r}\,dr_0 + \frac{1}{\pi}\int_0^b K_{21}(r_0,r) f_1(r_0)\,dr_0 = -u_{1r}^{0\prime}(r,\pi), \\
&0 \leqslant r \leqslant b
\end{aligned}
$$

for determining the abrupt change in the tangential stress.

When $\mu_0 = 0$, $\nu_0 = 0$ we obtain $f_1(r) = 0$ from (6) and an equation describing the elastic equilibrium of two half-planes of different kinds with a mathematical slit going from one of its vertices to the straight line of the bond /4/

$$\frac{a_{12}}{\pi}\int_0^b \frac{f_2(r_0)}{r_0 - r}\,dr_0 + \frac{1}{\pi}\int_0^b K_{12}(r_0, r)\, f_2(r_0)\,dr_0 = -\tau^0_{100}(r, \pi), \quad 0 < r < b$$

By means of the change of variables $2r = b(x+1)$ and $2r_0 = b(t+1)$ and taking account of the relationship

$$\int_{-1}^{x} \varphi(t)\,dt = \frac{1}{2}\int_{-1}^{1} \operatorname{sign}(x-t)\,\varphi(t)\,dt + \frac{1}{2}\int_{-1}^{1}\varphi(t)\,dt, \quad -1 < x < 1$$

the integral Eq.(6) and condition (7) are written in the form

$$\frac{\lambda_1}{\pi}\int_{-1}^{1}\frac{\varphi_2(t)}{t-x}\,dt + \lambda_2'\int_{-1}^{1}\operatorname{sign}(x-t)\,\varphi_2(t)\,dt + \frac{1}{\pi}\int_{-1}^{1} k(x,t)\,\varphi_2(t)\,dt = \Phi(x) \tag{10}$$

$$\varphi_1(x) = \mu_0\varphi_2(x), \quad -1 < x < 1$$

$$\int_{-1}^{1}\varphi_2(t)\,dt = A' \tag{11}$$

Here

$$k(x, t) = \sum_{n=0}^{2} d_n (1+x)^n \frac{d^n (t+x+2)^{-1}}{dx^n},$$

$$d_n = \mu_0 k_2 d_{11}{}^n + k_2 d_{12}{}^n + \mu_0 d_{21}{}^n + d_{22}{}^n$$

$$d_{ij}{}^0 = b_{ij}{}^0, \quad d_{ij}{}^1 = -b_{ij}{}^1, \quad d_{ij}{}^2 = {}^1/_2 b_{ij}{}^2 \quad (n = 0, 1, 2;\ i, j = 1, 2)$$

$$\Phi(x) = F(2r/b - 1) - \lambda_2' A, \quad \varphi_j(x) = f_j(2r/b - 1) \quad (j = 1, 2),$$

$$A' = 2A/b, \ \lambda_2' = b\lambda_2/4$$

The solution of the SIE under the condition (7) is sought in the form

$$\varphi_j(t) = (1-t)^\alpha (1+t)^\beta g_j(t), \quad -1 < \alpha, \ \beta < 0, \quad -1 < t < 1 \quad (j = 1, 2) \tag{12}$$

($g_j(t)$ is a bounded measurable function). Using the asymptotic values of the Cauchy integrals /12/

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\varphi_i(t)}{t-x}\,dt = -g_i(-1)\,2^\alpha \operatorname{ctg}\pi\beta\,(x+1)^\beta + g_i(1)\,2^\beta \operatorname{ctg}\pi\beta\,(1-x)^\alpha + \Phi_{0i}(x) \tag{13}$$

$$\frac{1}{\pi}\int_{-1}^{1}\varphi_i(t)(x+1)^n \frac{d^n(t+x+2)^{-1}}{dx^n}\,dt = -g_i(-1)\frac{2^\alpha}{\sin\pi\beta}\,\beta(\beta-1)\ldots(\beta-n+1)(x+1)^\beta + (x+1)^n\frac{d^n\Phi_{1i}(x)}{dx^n}, \quad -1 \leqslant x \leqslant 1 \quad (n = 0, 1, 2)$$

$$\lim_{x\to -1}(1+x)^{-\beta}[\Phi(x), \Phi_{0i}(x), (x+1)^n\Phi_{1i}^{(n)}(x)] = 0$$

$$\lim_{x\to 1}(1-x)^{-\alpha}[\Phi(x), \Phi_{0i}(x), (x+1)^n\Phi_{1i}^{(n)}(x)] = 0$$

where $\Phi_{ki}(x) \in H_{\gamma ki}([-1,1];\ R^1)$, γ_{ki} $(k = 0, 1;\ i = 1, 2)$ is the Hölder index, we obtain from the SIE (10)

$$\operatorname{ctg}\pi\alpha = 0, \ \alpha = -{}^1/_2$$
$$\lambda_1\cos\pi\beta + d_0 + d_1\beta + d_2\beta(\beta-1) = 0 \tag{14}$$

An investigation of the second equation in (14) showed that it contains one real root β_0 in the interval $(-1, 0)$ for arbitrary values of $m = \mu_2/\mu_1$ and $\varepsilon = \mu_0/\mu_1$. Values for

$-\beta_0 \cdot 10^3$ are presented in the table for $\nu_0 = \nu_1 = \nu_2 = 0.3$.

m	$\varepsilon = 10^{-2}$	0.1	10	10^2	m	$\varepsilon = 10^{-2}$	0.1	10	10^2
10^3	289	255	871	977	0.98	502	502	498	497
10^2	294	260	851	929	0.1	754	786	377	321
23.1	302	278	742	855	0.045	826	858	367	304
10	333	305	709	786	10^{-2}	915	933	360	292
1.02	498	497	501	502	10^{-3}	973	963	358	289
1.0	500	500	500	500					

Taking (12) into account the SIE (10) under the conditions (11) is solved numerically by applying the analogue of the Gauss-Jacobi quadrature fromula /4/ for the Cauchy integrals. We consequently arrive at a system of linear algebraic equations

$$\frac{1}{\pi}\sum_{j=1}^{N} g_2(t_j) W_j \left[\frac{\lambda_1}{t_j - x_k} + \lambda_2' \pi \operatorname{sign}(x_k - t_j) + k(x_k, t_j)\right] = \Phi(x_k) \tag{15}$$

$$\sum_{j=1}^{N} W_j g_2(t_j) = A', \quad k = 1, 2, \ldots, N-1$$

Here

$$P_N^{(\alpha,\beta)}(t_j) \equiv 0, \quad P_{N-1}^{(-\alpha,-\beta)}(x_k) \equiv 0$$

$$W_j = -\frac{2N+\alpha+\beta+2}{(N+1)!\,(N+\alpha+\beta+1)} \frac{\Gamma(N+\alpha+1)\,\Gamma(N+\beta+1)}{\Gamma(N+\alpha+\beta+1)} \times \frac{2^{\alpha+\beta}}{P_N^{(\alpha,\beta)\prime}(t_j)\, P_{N+1}^{(\alpha,\beta)}(t_j)}$$

$P_N^{(\alpha,\beta)}(\cdot)$ are Jacobi polynomials, and $\Gamma(\cdot)$ is the gamma function. The stress intensity factors at the inclusion endfaces are determined in the form

$$k(0) = \lim_{r\to 0} 2^{1/2} r^{-\beta} \tau_{2\theta\theta}(r, 0), \quad k(b) = \lim_{r\to b} [2(r-b)]^{1/2} \tau_{1\theta\theta}(r, \pi) \tag{16}$$

where the stresses $\tau_{1\theta\theta}(r, \pi)$ are given by the relationships (2) while $\tau_{2\theta\theta}(r, 0)$ on the basis of /8/ are the following:

$$\tau_{2\theta\theta}(r, 0) = \tau_{2\theta\theta}^0(r, 0) + H(r), \quad 0 < r < \infty \tag{17}$$

$$H(r) = \frac{1}{\pi}\int_0^b \sum_{j=1}^{2} Q_j(r_0, r) f_j(r_0)\, dr_0$$

$$Q_j(r_0, r) = \sum_{n=0}^{1} q_j^n \frac{r^n}{(r_0 + r)^{n+1}} \quad (j = 1, 2)$$

$$q_1^0 = {}^1/_2 m\,[(2 + \varkappa_1)\, l_3 - 3l_4], \quad q_1^1 = m\,(l_4 - l_3)$$

$$q_2^0 = \mu_2\,(3l_4 - l_3), \quad q_2^1 = 2\mu_2\,(l_3 - l_4)$$

$$l_3 = (1 + m\varkappa_1)^{-1}, \quad l_4 = (m + \varkappa_2)^{-1}$$

Using the asymptotic relationships (13), we obtain from (17)

$$\tau_{2\theta\theta}(r, 0)\big|_{r>0} = -\mu_1^* f_1(r)\big|_{r<0} + \mu_2^* f_2(r)\big|_{r<0} + O(r^\lambda), \quad \lambda > 0 \tag{18}$$

$$\mu_1^* = m\,\frac{(3+2\beta)\, l_4 - (2 + \varkappa_1 + 2\beta)\, l_3}{2\sin\pi\beta}$$

$$\mu_2^* = \mu_2\,\frac{(3+2\beta)\, l_4 - (1+2\beta)\, l_3}{\sin\pi\beta}$$

From (2) we obtain at the endface $r = b$

$$\tau_{1\theta\theta}(r, \pi)\big|_{r>b} = a_{11} f_1(r)\big|_{r<b} + a_{12} f_2(r)\big|_{r<b} + O[(r-b)^\lambda], \quad \lambda > 0 \tag{19}$$

Substituting (18) and (19) into (16) yields

$$k(0) = a_a^{-\beta}\,[\mu_1^* q_1(-1) + \mu_2^* g_2(-1)] \tag{20}$$

$$k(b) = a_0^{1/2} 2^{1/2+\beta}\,[a_{11} g_1(1) + a_{12} g_2(1)]$$

The dependence of the stress intensity factors $k(0)/(\sigma_j a_0^{-\beta})$ (the dashed curve) and $k(b)/(\sigma_j a_0^{1/2})$ (the solid curve) $(j = 1, 2)$, calculated by means of (20), on the relative stiffness of the inclusion ε is shown in Fig.2. The subscript 1 and curves 1 correspond to the cases $m = 23.08$ (aluminium epoxy), $j = 2$ and curves 2 to $m = 0.045$ (epoxy-aluminium). The external tensile load (taking (1) into account) was given as

$$\sigma_0 = k_1^1/k_0^1 = 0.428,\quad \sigma_1 = 1.0,\quad \sigma_2 = 19.0 \text{ (aluminium-epoxy)}$$
$$\sigma_0 = k_1^2/k_0^1 = 0.428,\quad \sigma_2 = 1.0,\quad \sigma_1 = 18.3 \text{ (epoxy-aluminium)} \qquad (21)$$
$$\varkappa_j = 3 - 4\nu_j,\ \nu_0 = \nu_1 = \nu_2 = 0.3,\quad h/b = 0,1$$

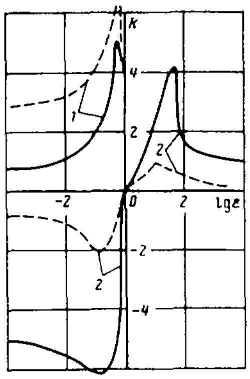

Fig.2

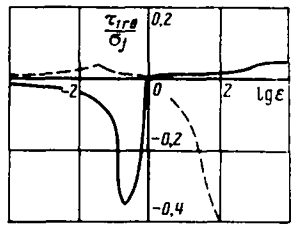

Fig.3

The values of $g_j(\pm 1)$ $(i = 1, 2)$ are determined from the data of the solution of the system of algebraic Eqs.(15) $g_i(t_j)$ $(i = 1, 2)$ using the interpolation formula /13/

$$g_i(x) \simeq \sum_{k=0}^{N-1} e_{ki} P^{(\alpha,\beta)}(x), \quad e_{ki} = p_k^{-1} \sum_{j=1}^{N} W_j P_k^{(\alpha,\beta)}(t_j)\, g_i(t_j)$$
$$p_k = \frac{2^{\alpha+\beta+1}}{2k+\alpha+\beta+1}\, \frac{\Gamma(k+\alpha+1)\,\Gamma(k+\beta+1)}{k!\,\Gamma(k+\alpha+\beta+1)}$$

It is necessary to take $N = 25$ to achieve a 1% relative accuracy of the calculation (determined by comparing the next approximations with the preceding ones). A change in the shear stresses $\tau_{1r\theta}(r,\pi)/\sigma_j$ $(j = 1, 2)$ at the point $r = 0.1\,b$ of the axial line of the inclusion is shown in Fig.3 as a function of ε The tensile stresses were also given in the form (21).

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A VARIATIONAL METHOD OF SOLVING AN ELASTIC-PLASTIC PROBLEM FOR A BODY WITH A CIRCULAR HOLE*

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An approach based on the theory of variational inequalities and a generalized plastic analogy for the solution of the elastic-plastic problem (EPP) concerning the state of stress of a body weakened by a circular hole without the assumption regarding total enclosure of the hole by a plastic zone is proposed. The Haar-Karman hypothesis or an equivalent assertion is not used here. Generalizations are given to the case of a plastic inhomogeneous body and for the utilization of an exponential flow condition. Examples are considered and a simple method is proposed for estimating the plastic zone dimensions.

It was assumed in the well-known solution given by Galin /1/ of the EPP on the biaxial tension of a plane with a circular hole and its generalizations /2-5/ that the plastic domain completely encloses the hole. The majority of existing solutions have been obtained for the stress concentration around a hole in an infinite domain.

Let us consider the problem of the plane strain of a body Ω with smooth outer contour L and a circular hole C of radius a (Fig.1). Near the outer boundary the medium under the loads acting on the body is in an elastic state. We shall also assume that if the plastic zone does not enclose the hole, then all its connected subdomains lie within appropriate characteristic triangles such that, as in the case of total enclosure, the stresses in the plastically homogeneous zone D^p are described by the relationships (tensile conditions)

$$\sigma_{rr}{}^p = 2\tau_s \ln (r/a),\ \ \sigma_{\vartheta\vartheta}^p = 2\tau_s [1 + \ln (r/a)],\ \ \sigma_{r\vartheta}^p = 0 \tag{1}$$

where r, ϑ are polar coordinates connected to the centre of the hole and τ_s is the plasticity limit.

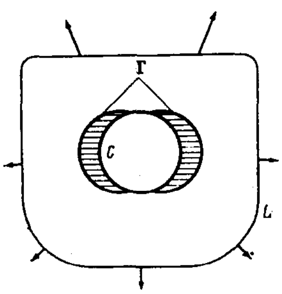

Fig.1

It is convenient to formulate a statically determinable EPP in the terminology of the Airy stress function /1-3/: it is required to find the function $u(x, y)$ which satisfies the biharmonic equation in the elastic zone D^e

$$\Delta^2 u = 0 \tag{2}$$

and the condition

$$\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right)^2 + 4\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 < 4\tau_s^2 \tag{3}$$

and satisfies the following equation in the plastic zone D^p:

$$\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right)^2 + 4\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = 4\tau_s^2 \tag{4}$$

The conditions

$$u|_L = f(l),\ \ \left.\frac{\partial u}{\partial n}\right|_L = f_1(l),\ \ u|_C = 0,\ \ \left.\frac{\partial u}{\partial n}\right|_C = 0 \tag{5}$$

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